

VERTEX ISOPERIMETRY AND INDEPENDENT SET STABILITY FOR TENSOR POWERS OF CLIQUES

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ABSTRACT. The tensor power of the clique on t vertices (denoted by K_t^n) is the graph on vertex set $\{1, \dots, t\}^n$ such that two vertices $x, y \in \{1, \dots, t\}^n$ are connected if and only if $x_i \neq y_i$ for all $i \in \{1, \dots, n\}$. Let the density of a subset S of K_t^n to be $\mu(S) := \frac{|S|}{t^n}$, and let the vertex boundary of a set S to be vertices which are incident to some vertex of S , perhaps including points of S . We investigate two similar problems on such graphs.

First, we study the vertex isoperimetry problem. Given a density $\nu \in [0, 1]$ what is the smallest possible density of the vertex boundary of a subset of K_t^n of density ν ? Let $\Phi_t(\nu)$ be the infimum of these minimum densities as $n \rightarrow \infty$. We find a recursive relation allows one to compute $\Phi_t(\nu)$ in time polynomial to the number of desired bits of precision.

Second, we study given an independent set $I \subseteq K_t^n$ of density $\mu(I) = \frac{1}{t}(1 - \epsilon)$, how close it is to a maximum-sized independent set J of density $\frac{1}{t}$. We show that this deviation (measured by $\mu(I \setminus J)$) is at most $4\epsilon^{\frac{\log t}{\log t - \log(t-1)}}$ as long as $\epsilon < 1 - \frac{3}{t} + \frac{2}{t^2}$. This substantially improves on results of Alon, Dinur, Friedgut, and Sudakov (2004) and Ghandehari and Hatami (2008) which had an $O(\epsilon)$ upper bound. We also show the exponent $\frac{\log t}{\log t - \log(t-1)}$ is optimal assuming n tending to infinity and ϵ tending to 0. The methods have similarity to recent work by Ellis, Keller, and Lifshitz (2016) in the context of Kneser graphs and other settings.

The author hopes that these results have potential applications in hardness of approximation, particularly in approximate graph coloring and independent set problems.

1. INTRODUCTION

1.1. Vertex isoperimetry. For any undirected graph $G = (V_G, E_G)$ and $S \subseteq V_G$, we define the *vertex boundary* of S to be

$$\partial S := \{x \in V_G : \text{exists } y \in S \text{ such that } \{x, y\} \in E_G\}.$$

Furthermore, we define the *density* of S to be

$$\mu(S) := \frac{|S|}{|V_G|}.$$

The relationship between $\mu(S)$ and $\mu(\partial S)$, particularly when $\mu(S)$ is sufficiently small (typically at most $1/2$). Is known as a *vertex isoperimetric inequality*. Such relationships are captured by the *isoperimetric parameter* (or *isoperimetric profile*) of a graph

$$\Phi(G, \nu) = \inf\{\mu(\partial S) : \mu(S) \geq \nu\}$$

Proving such inequalities for various graphs is a frequent topic in the literature (e.g., [BHT00, CEK13]). Typically such works focus on a *linear* or near-linear relationship between $\mu(\partial S)$ and $\mu(S)$, known as the *isoperimetric constant*.

$$(1) \quad h(G) = \inf \left\{ \frac{\mu(\partial S)}{\mu(S)} \mid S \subset V_G, \mu(S) \in (0, 1/2] \right\}.$$

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In this paper, we study graphs for which there is an order-of-magnitude difference between $\mu(S)$ and $\mu(\partial S)$, when $\mu(S)$ is sufficiently small. For example, if $\mu(\partial S) \geq \sqrt{\mu(S)}$ for all S , we would like to say that G expands by a power of 2. Such ‘hyper-expansion’ can be captured by what we coin as the *isoperimetric exponent*. For all $\epsilon > 0$ consider.

$$(2) \quad \eta(G, \epsilon) = \inf \left\{ \frac{\log \mu(S)}{\log \mu(\partial S)} \mid S \subset V_G, \mu(\partial S) \in (0, \epsilon) \right\}$$

where \log is the natural logarithm. In other words, for every subset S of G of density δ , the boundary of S has density at least $\delta^{1/\eta(G, \epsilon)}$. The larger the parameter $\eta(G)$ is, the more ‘expansive’ the graph is. It is easy to see that $\eta(G, \epsilon)$ is in general a decreasing function of ϵ . As we often work with large subsets of our graph, we let $\eta(G) := \eta(G, 1)$.

In this paper, we study the isoperimetric profile of the tensor powers of cliques. For undirected graphs $G = (V_G, E_G), H = (V_H, E_H)$, we define the *tensor product* $G \otimes H$ to be the undirected graph on vertex set $V_1 \times V_2$ such that an edge connects (u_1, v_1) and (u_2, v_2) if and only if $\{u_1, u_2\} \in E_G$, and $\{v_1, v_2\} \in E_H$. Note that up to isomorphism, the tensor product is both commutative and associative. We then denote $\otimes^n G$ to be the tensor product of n copies of G . Since this is the only graph product discussed in this article, we shorten this to G^n . In this article, we focus on the case that $G = K_t$, where K_t is the complete graph on $t \geq 3$ vertices. It turns out for such graphs that for all $\epsilon > \frac{1}{t^n}$, $\eta(G) = \eta(G, \epsilon)$.

In particular, we shall compute the following.

Theorem 1. *For all $t \geq 3$ and all positive integers n ,*

$$(3) \quad \eta(K_t^n) = \eta(K_t) = \frac{\log t}{\log t - \log(t-1)} = t \log t + \Theta(\log t).$$

In addition to this high-level structure, we give a more-fine-tuned analysis of the behavior of $\Phi_t(\eta) := \inf_{n \geq 1} \Phi(K_t^n, \eta)$. (See Theorem 10.)

1.2. Independent set stability. With these vertex isoperimetric inequalities, we apply them to the understanding the structure of near-maximum independent sets of graphs. Such results are known as *stability* results.

Such results are not just of interest within combinatorics, a better understanding of independent set stability of certain graphs, such as K_t^n , have resulted in advances in hardness of approximation, particularly in construct dictatorship tests for approximate graph coloring and independent set problems (e.g., [ADFS04, DFR08, BG16]). In fact the investigation which led to the results in this paper was inspired by the pursuit of such results.

A landmark result of this form due to [ADFS04] is as follows.

Theorem 2 ([ADFS04]). *For all $t \geq 3$ there exist C_t with the following property. For any positive integer n , Let $I \subset K_t^n$ be an independent set such that $\epsilon = 1 - t\mu(I)$, then there exists an independent set $J \subset K_t^n$ of maximum size ($\mu(J) = 1/t$) such that $\mu(I \Delta J) \leq C_t \epsilon$, where $S \Delta T = (S \setminus T) \cup (T \setminus S)$.*

In other words, independent sets of near-maximum size are similar in structure to the maximum independent sets. Note that if J is an independent set of maximum size, then for some $i \in [n]$ and $j \in [t]$, we have that

$$J = [t]^{i-1} \times \{j\} \times [t]^{n-i}.$$

This is a well-known result due to [GL74] (see [AS04] for a proof using Fourier analysis).

Ghandehari and Hatami improved this result (Theorem 1 of [GH08]) to show that if $t \geq 20$ and $\epsilon \leq 10^{-9}$ then C_t can be replaced with $40/t$. Both results were proven using Fourier analysis.

We improve upon this result in two steps. First, with an application of Theorem 1 we improve Theorem 2 in a black-box matter to obtain.

Theorem 3. *For all $t \geq 3$, there exists $\epsilon_t > 0$ with the following property. For any positive integer n , Let $I \subset K_t^n$ be an independent set such that $\epsilon = 1 - t\mu(I) < \epsilon_t$, then there exists an independent set $J \subset K_t^n$ of maximum size ($\mu(J) = 1/t$) such that*

$$(4) \quad \mu(I \setminus J) \leq 4\epsilon^{\eta(K_t)} = 4\epsilon^{\log t / (\log t - \log(t-1))}.$$

Remark 1. Since $\mu(I \setminus J) \leq 4\epsilon^{\eta(K_t)}$,

$$\mu(I \Delta J) = \mu(I \setminus J) + \mu(J \setminus I) = \mu(J) - \mu(I) + 2\mu(I \setminus J) = \frac{\epsilon}{t} + 4\epsilon^{\eta(K_t)},$$

so our result gives the optimal first-order structure for Theorem 2 assuming ϵ is sufficiently small. Furthermore, in Appendix C, we give examples of independent sets of K_t^n with arbitrarily small density (assuming $n \rightarrow \infty$) for which the exponent $\eta(K_t)$ is optimal.

Next, using a purely combinatorial argument we pin down a precise value for ϵ_t .

Theorem 4. *In Theorem 3, for all $t \geq 3$, one may set $\epsilon_t = 1 - \frac{3}{t} + \frac{2}{t^2}$. In other words, the theorem applies for all independent sets I such that $\mu(I) > \frac{3t-2}{t^3}$.*

The choice of ϵ_t is not arbitrary, it corresponds to the density of the following independent set.

$$I = \{(1, 1, a), (1, a, 1), (a, 1, 1) : a \in [t]\} \times [t]^{n-3}.$$

Note that $\mu(I) = \frac{3t-2}{t^3}$. This set represents a phase transition in the independent sets from ‘dictators’ to ‘juntas,’ as the I constructed above is equally influenced by 3 coordinates (where ‘influence’ is in the sense of [ADFS04]). Such phase transitions have been studied in the literature [DFR08], but this may be the first work to highlight the exact transition point.

Additionally, to the best of the author’s knowledge, this is the first known purely combinatorial proof of Theorem 2.

1.2.1. Related work. Such stability results for independent sets have also been studied for Kneser graphs. A result similar to that of Theorem 2 was proved by [Fri08]. Numerous other works in the literature [DF09, DS05, BM08, Kee08, KM10, FKMW16, FM16] prove generalized stability results for Kneser graphs or other structures related to intersecting families.

A result which also finds a “tight” super constant exponent $\eta > 1$ for the independent set stability is proved in some very recent work [EKL16b, EKL16a, EL16, KL16b, KL16a, EKL17] on Kneser graphs and related structures. (See also [EKN17] and Proposition 4.3 of [Fil16].) The techniques have high-level similarity to the ones adopted here:¹ particularly in their use of compressions to prove a isoperimetric inequality which they then bootstrap to a combinatorial independent set stability result.

1.3. Paper organization. In Section 2 we prove the claimed vertex isoperimetric inequalities. In Section 3, we prove the stability results for near-maximum independent sets in K_t^n . Appendix A proves some algebraic inequalities omitted from the main text. Appendix B proves Theorem 10, which gives a refined understanding the isoperimetric profile of Kneser graphs. Appendix C shows that the exponent of $\eta(t)$ in Theorems 3 and 4 is optimal.

2. VERTEX ISOPERIMETRIC INEQUALITIES

In this section, we proceed to prove the isoperimetry results claimed in Section 1.1.

Identify the vertex set of K_t^n with $[t]^n$. Two vertices of $x, y \in [t]^n$ are connected in K_t^n if and only if $x_i \neq y_i$ for all $i \in [n]$. Denote $y_{-i} := (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$. We often write y as (y_i, y_{-i}) when it is clear from context which coordinate is being inserted.

¹The author became aware of these similar proofs only after writing major portions of the manuscript.

2.1. Compressions. A useful tool in our study will be the operation of the well-known technique of *compressions* (e.g., [Sau72, She72]). Although compressions are not strictly necessary to prove Theorem 1, they are essential in the proof of stronger isoperimetry results as well as Theorem 4, so we introduce the machinery now.

For $S \subseteq [t]^n$ be a subset, define the *compression of S in coordinate i* to be

$$(5) \quad c_i(S) = \{x \in [t]^n : x_i \leq |\{y \in S : y_{-i} = x_{-i}\}|\}.$$

Informally, we ‘shift’ each element of S to be as small as possible in the i th direction. Note that $\mu(c_i(S)) = \mu(S)$ for all $S \subseteq [t]^n$. It is easy to see that c_i is *nilpotent*: $c_i(c_i(S)) = c_i(S)$ for all $S \subseteq [t]^n$ and $i \in [n]$.

We say that a set S is *compressed* if $c_i(S) = S$ for all $i \in [n]$. Equivalently, for all $x \in S$ there is no $y \in [t]^n \setminus S$ such that $x_i \leq y_i$ for all $i \in [n]$.

Remark 2. Note that every time a compression c_i is applied, the quantity

$$\Sigma(S) := \sum_{x \in S} \sum_{j \in [n]} x_j$$

decreases or stays the same (in which case $c_i(S) = S$). Thus, since $\Sigma(S)$ is always positive, there must exist a finite sequence of compressions which can be applied to S to make the set compressed.

Now we show that compressions respect independent sets of K_t^n . This result is not needed until Section 3, but the proof does give intuition for how the compressions work.

Claim 5. *For all $i \in [n]$ and all $I \subset [t]^n$ independent set of K_t^n , $c_i(I)$ is also an independent set of K_t^n .*

Proof. Assume not, then there exist $x, y \in c_i(I)$ such that $\{x, y\}$ is an edge. In particular, since $x_i \neq y_i$, we must have that $x_i \neq 1$ or $y_i \neq 1$. Assume without loss of generality that $y_i \neq 1$. Then, by definition of $c_i(I)$, there must be $z := (1, y_{-i}) \in c_i(I)$. Since $x, y, z \in c_i(I)$, there must be $x', y', z' \in I$ such that

$$\begin{aligned} x_{-i} &= x'_{-i} \\ y_{-i} &= z_{-i} = y'_{-i} = z'_{-i} \\ y'_i &\neq z'_i. \end{aligned}$$

Since $y'_i \neq z'_i$, we must either have that $x'_i \neq y'_i$ or $x'_i \neq z'_i$. In the former case, $\{x', y'\}$ is an edge of K_t^n and in the latter case $\{x', z'\}$ is an edge of K_t^n . This contradicts the fact that I is an independent set. \square

Next we show that compressions can only decrease the size of the vertex boundary.

Claim 6. *For all $i \in [n]$ and $S \subseteq [t]^n$, $|\partial c_i(S)| \leq |\partial S|$.*

Proof. Fix $\bar{a} := a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in [t]$. Consider $T = \{(a_1, \dots, a_{i-1})\} \times [t] \times \{(a_{i+1}, \dots, a_n)\} \subset [t]^n$.

Note that for every vertex $v \in [t]^n$, $\partial\{v\} \cap T$ either has 0 or $t-1$ elements. Thus, $|T \cap \partial S| \in \{0, t-1, t\}$. We claim that $|T \cap \partial c_i(S)| \leq |T \cap \partial S|$ for all T .

- If $|T \cap \partial S| = 0$, then there are no edges between S and T and shifting the vertices of S in the i th coordinate cannot change that. Thus, $|T \cap \partial c_i(S)| = 0$.
- If $|T \cap \partial S| = t-1$, then the set $\partial T \cap S$ must be constant in the i th coordinate. Thus, $c_i(\partial T \cap S) = \partial T \cap c_i(S)$ is constant in the i th coordinate, so $|T \cap \partial c_i(S)| = t-1$.
- If $|T \cap \partial S| = t$, then trivially $|T \cap \partial c_i(S)| \leq t$.

Thus, summing $|T \cap \partial c_i(S)| \leq |T \cap \partial S|$ across all possible T , we have that $|\partial c_i(S)| \leq |\partial S|$. \square

Remark 3. The proof crucially uses the fact that ∂S can include elements of S . If we instead had defined the vertex boundary to be $\partial S \setminus S$, there is a simple counterexample. Consider $t = 3$ and $n = 2$ and $S = \{(1, 2), (1, 3), (2, 1), (3, 1)\}$. Then it is not hard to check that $|\partial S| = |\partial c_1(S)| = 8$, but $|\partial S \setminus S| = 4 < 5 = |\partial c_1(S) \setminus c_1(S)|$.

2.2. Proof of Theorem 1. Define

$$(6) \quad \eta(t) := \frac{\log t}{\log t - \log(t-1)} = t \log t + \Theta(\log t).$$

First, we show that $\eta(K_t^n) \leq \eta(t)$. In fact, we show a whole family of equality cases.

Claim 7. *For all positive integers n and t such that $t \geq 3$, $\eta(K_t^n) \leq \eta(t)$.*

Proof. For all integers $k \in [n]$, consider $S = \{1\}^k \times [t]^{n-k}$. Then $\partial S = \{2, \dots, t\}^k \times [t]^{n-k}$. Thus,

$$\eta(K_t^n) \leq \frac{\log \mu(S)}{\log \mu(\partial S)} = \frac{\log t^{-k}}{\log((t-1)^{kt-k})} = \frac{k \log \frac{1}{t}}{k \log \frac{t-1}{t}} = \eta(t). \quad \square$$

The lower-bound is more difficult, we first need the following inequality, proved in Appendix A.

Claim 8. *Let $t \geq 2$ be a positive integer and let $x \geq y \geq 0$ be real numbers, then*

$$(7) \quad y^{1/\eta(t)} + (t-1)x^{1/\eta(t)} \geq (t-1)(x + (t-1)y)^{1/\eta(t)}$$

Lemma 9. *For positive integers $n \geq 1$ and $t \geq 3$ and all $S \subseteq [t]^n$, we have that*

$$(8) \quad \mu(\partial(S)) \geq \mu(S)^{1/\eta(t)}.$$

Therefore $\eta(K_t^n) \geq \eta(t)$.

Proof. By Claim 6 and Remark 2, it suffices to consider the case that S is compressed. We now proceed by induction on n .

For our base case, $n = 1$, we must have that $S = \emptyset$ in which case (8) is trivial, or $S = [k]$ for some positive integer $k \leq t$. If $S = [1]$, then $\partial S = \{2, \dots, t\}$, in which case we have an equality case of (8) by the proof of Claim 7. Otherwise, if $k \geq 2$, then $\partial S = [t]$, so $\mu(\partial S) = 1$, so (8) holds.

For $n \geq 2$, assume by the induction hypothesis that (8) is true for all $S \subseteq \mathbb{Z}_t^m$ where $1 \leq m < n$. For all $i \in [t]$, let

$$(9) \quad S_i := \{x_{\neg n} : x_n \in S, x_n = i\}$$

$$(10) \quad (\partial S)_i := \{x_{\neg n} : x_n \in \partial S, x_n = i\}.$$

Since S is compressed for all $1 \leq i \leq j \leq t$, we have that $S_i \supseteq S_j$. Thus, if $i \in \{2, \dots, t\}$ is nonzero, for any $x \in (\partial S)_i$, there is $y \in S_0$ connected to x by an edge of K_t^{n-1} . Thus, $\partial S_0 \subseteq (\partial S)_i$. Similarly, for any $x \in (\partial S)_0$, there is $y \in S_1$ such that x is disjoint from y . Therefore, $\partial S_1 \subseteq (\partial S)_0$. Putting these together,

$$\begin{aligned} \mu(\partial S) &= \frac{1}{t} \sum_{i \in [t]} \mu((\partial S)_i) \\ &\geq \frac{1}{t} (\mu(\partial S_1) + (t-1)\mu(\partial S_0)) \\ &\geq \frac{1}{t} \left(\mu(S_1)^{1/\eta(t)} + (t-1)\mu(S_0)^{1/\eta(t)} \right), \end{aligned}$$

where we applied the inductive hypothesis in the last step. Applying Claim 8, using the fact that $0 \leq \mu(S_1) \leq \mu(S_0)$, we have that

$$\begin{aligned}
\mu(\partial(S)) &\geq \frac{1}{t} \left(\mu(S_1)^{1/\eta(t)} + (t-1)\mu(S_0)^{1/\eta(t)} \right) \\
&\geq \frac{t-1}{t} (\mu(S_0) + \mu(S_1))^{1/\eta(t)} \\
&\geq \frac{t-1}{t} \left(\sum_{i \in [t]} \mu(S_i) \right)^{1/\eta(t)} \\
&= \left(\frac{1}{t} \sum_{i \in [t]} \mu(S_i) \right)^{1/\eta(t)} \\
&= \mu(S)^{1/\eta(t)},
\end{aligned}$$

as desired. \square

Claim 7 and Lemma 9 together imply Theorem 1.

2.3. A fine-tuned understanding of the isoperimetric profile. Recall the (vertex) isoperimetric profile of a graph G to be

$$\Phi(G, \nu) := \inf \{ \mu(\partial S) : \mu(S) \geq \nu \}.$$

For $t \geq 3$ fixed, define

$$\Phi_t(\nu) := \inf_{n \geq 1} \Phi(K_t^n, \nu).$$

Note that Φ_t is non-decreasing. It is easier to work with $\Phi_t(\nu)$ instead of each $\Phi(K_t^n, \nu)$ directly to avoid complications with the discrete behavior of $\Phi(K_t^n, \nu)$ when n is small. By Theorem 1,

$$(11) \quad \Phi_t(\nu) \geq \nu^{1/\eta(t)}.$$

This is tight whenever $\nu = t^{-k}$ for any integer $k \geq 0$, but ceases to be tight when $\log_t(\nu)$ is non-integral (see Figure 1).

The following recursive relationship allows one to compute $\Phi_t(\nu)$ to arbitrary precision.

Theorem 10. *For all $t \geq 3$,*

$$(12) \quad \Phi_t(\nu) = \begin{cases} \frac{t-1}{t} \Phi_t(t\nu) & \nu < 1/t \\ \frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{t\nu-1}{t-1}\right) & \nu \geq 1/t \end{cases}.$$

Using the simple fact that $\Phi_t(0) = 0$ and $\Phi_t(1) = 1$, the above equation is extremely powerful. For example,

$$\Phi_3\left(\frac{5}{9}\right) = \frac{2}{3} + \frac{1}{3} \Phi_3\left(\frac{1}{3}\right) = \frac{8}{9},$$

which is an exact bound compared to $(\frac{5}{9})^{1/\eta(3)} \approx \frac{7.24}{9}$. This recursion is what allowed the creation of Figure 1.

Theorem 10 is proved in Appendix B. This more refined understanding of Φ_t proves critical in the combinatorial proof of Theorem 4.

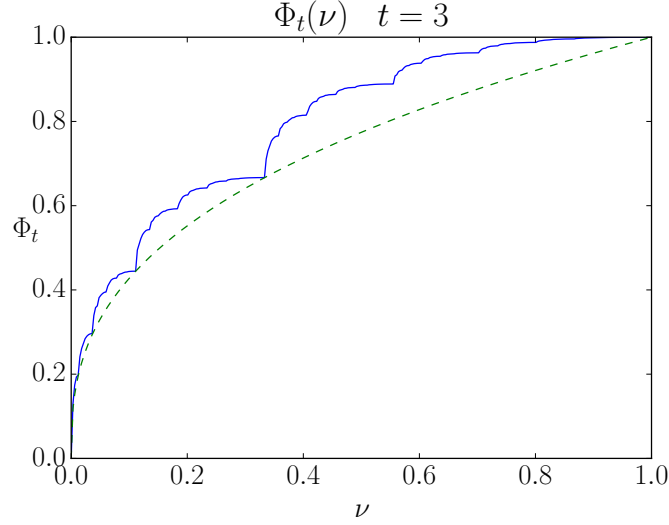


FIGURE 1. A graph of $\Phi_t(\nu)$ for $t = 3$. The dashed curve $\nu^{1/\eta(t)}$ is for reference.

3. INDEPENDENT SET STABILITY RESULTS

3.1. Black-box result for clique tensor powers. First, we show that if a large independent set I is somewhat close to a maximum-sized independent set J , then it is really close to J . We fix positive integers n and $t \geq 3$.

Lemma 11. *Let $I \subset [t]^n$ be an independent set with $\epsilon := 1 - t\mu(I)$. Assume there exists a maximum-sized independent set J such that*

$$\mu(I \setminus J) < \frac{1}{t^3}.$$

Then,

$$\mu(I \setminus J) < 4\epsilon^{\eta(t)}.$$

Proof. Without loss of generality, we may assume that $J = [t]^{n-1} \times [1]$. Pick $J' = [t]^{n-1} \times \{j\}$ such that $j \neq 1$ but otherwise $\mu(I \cap J')$ is maximal. Let $\delta := \mu(I \setminus J)$. Since J and J' are disjoint, we have that

$$\mu(I \cap J') \geq \frac{\mu(I \setminus J)}{t-1} = \frac{\delta}{t-1}.$$

Now, consider $S = \partial(I \cap J')$. Recall the definition of $S_k \subseteq [t]^{n-1}$ from (9). Since $I \cap J' \subseteq J$ has the property that every element has the same last coordinate, $S_k = S_{k'}$ for all $k, k' \neq j$ and $S_j = \emptyset$. Thus, $\mu(S_k) = \frac{t}{t-1}\mu(S)$ for all $k \neq j$. Therefore,

$$\mu(S \cap J) = \frac{1}{t}\mu((S \cap J)_i) = \frac{1}{t}\mu(S_i) = \frac{1}{t-1}\mu(S).$$

Applying Theorem 1, we get that

$$\mu(S \cap J) = \frac{1}{t-1}\mu(\partial(I \cap J')) \geq \frac{1}{t-1}\mu(I \cap J')^{1/\eta(t)} \geq \frac{1}{t-1} \left(\frac{\delta}{t-1} \right)^{1/\eta(t)}.$$

Since I is an independent set, ∂I is disjoint from I . Since $S \cap J = \partial(I \cap J') \cap J \subseteq \partial I$, we have that $I \cap J$ and $S \cap J$ are disjoint. Therefore,

$$(13) \quad \mu(I \cap J) \leq \mu(J) - \mu(S \cap J) \leq \frac{1}{t} - \frac{1}{t-1} \left(\frac{\delta}{t-1} \right)^{1/\eta(t)}.$$

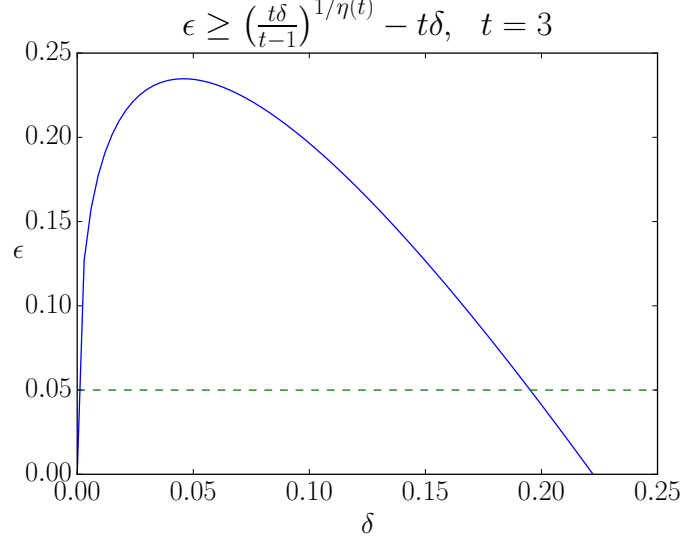


FIGURE 2. Plot of (15) when $t = 3$. Notice the bifurcation of solutions to (15) for a fixed ϵ (line $\epsilon = 0.05$ is dashed).

But, we also know that

$$(14) \quad \mu(I \cap J) = \mu(I) - \mu(I \setminus J) = \frac{1}{t}(1 - \epsilon) - \delta.$$

By (13) and (14)

$$\frac{1}{t}(1 - \epsilon) - \delta \leq \frac{1}{t} - \frac{1}{t-1} \left(\frac{\delta}{t-1} \right)^{1/\eta(t)} = \frac{1}{t} - \frac{1}{t} \left(\frac{t\delta}{t-1} \right)^{1/\eta(t)}.$$

Thus,

$$(15) \quad \epsilon \geq \left(\frac{t\delta}{t-1} \right)^{1/\eta(t)} - t\delta \geq \delta^{1/\eta(t)} - t\delta.$$

Consider Figure 2 which has a plot of the RHS of (15) when $t = 3$. If ϵ is sufficiently small, then the inequality holds only when δ is very small (polynomial in ϵ) or very large (about $\frac{1}{t}$). Since is ‘moderately’ small ($\delta \leq \frac{1}{t^3}$), we must have that δ is very small. Quantitatively, note that

$$\begin{aligned} t\delta &= t\delta^{1/\eta(t)}\delta^{1-1/\eta(t)} \\ &\leq t\delta^{1/\eta(t)} \left(\frac{1}{t^3} \right)^{1-1/\eta(t)} \\ &= t\delta^{1/\eta(t)} \frac{1}{t^3} \left(\frac{t^3}{(t-1)^3} \right) \\ &\leq \frac{t\delta^{1/\eta(t)}}{(t-1)^3}. \end{aligned}$$

So

$$\epsilon \geq \delta^{1/\eta(t)} \left(1 - \frac{t}{(t-1)^3} \right).$$

Therefore,

$$\delta \leq \left(\frac{(t-1)^3}{(t-1)^3 - t} \right)^{\eta(t)} \epsilon^{\eta(t)} \leq 4\epsilon^{\eta(t)},$$

where the last inequality follows from the following claim which is proved in Appendix A.

Claim 12. *For all $t \geq 3$,*

$$\left(\frac{(t-1)^3}{(t-1)^3 - t} \right)^{\eta(t)} \leq 4.$$

□

We now use this lemma to ‘amplify’ Theorem 2 to prove Theorem 3.

Proof of Theorem 3. Set $\epsilon_t := \frac{1}{C_t t^3} > 0$. Consider any independent set I of K_t^n such that $\epsilon := 1 - t\mu(I) < \epsilon_t$. Pick any maximum-sized J guaranteed by Theorem 2 such that

$$(16) \quad \delta := \mu(I \setminus J) \leq \mu(I \Delta J) \leq C_t \epsilon < \frac{1}{t^3}.$$

By Lemma 11, we have that

$$\delta \leq 4\epsilon^{\eta(t)},$$

as desired. □

3.2. Improved stability result for clique tensor powers. In this section we improve ϵ_t in Theorem 3 to an explicit expression. In fact, we may show that

$$\epsilon_t = 1 - \frac{3}{t} + \frac{2}{t^2}$$

which corresponds to independent sets I for which $\mu(I) > \frac{3t-2}{t^3}$.

First, we try to show that if an independent set I is large enough, then I is either very close to or very far from a maximum-sized independent set. To do this, we show that if I is ‘moderately far’ from a maximum-sized independent set, then this moderate-sized portion which is not in the maximum-sized independent set has such a large vertex boundary that it precludes a large portion of the maximum-sized independent set from being part of I , forcing the density of I to be at or below our threshold of $\frac{3t-2}{t^3}$.

We need a notation for the maximum sized independent sets. For all $i \in [t]$ and $j \in [n]$ let

$$(17) \quad J_{i,j} = [t]^{j-1} \times \{i\} \times [t]^{n-j}.$$

We say that I is *sorted* if there exists that for all $i_1, i_2 \in [t]$ and $j \in [n]$ we have that $i_1 \leq i_2$ implies that

$$\mu(I \cap J_{i_1,j}) \leq \mu(I \cap J_{i_2,j}).$$

Note that unlike compressions, we may assume without loss of generality that I is sorted since permuting the labels so that an independent set is sorted does not change its intersection sizes with the maximum independent sets.

Claim 13. *Let $I \subset [t]^n$ be a sorted independent set such that $\mu(I) > \frac{3t-2}{t^3}$ (or $1 - t\mu(I) < \epsilon_t$), then for all $j \in [n]$,*

$$(18) \quad \mu(I \setminus J_{1,j}) < \frac{t-1}{t^4} \text{ or } \mu(I \setminus J_{1,j}) > \frac{t-1}{t^3}.$$

Proof. Without loss of generality, we may let $j = n$. Denote $J := J_{1,j}$. Let $\delta = \mu(I \setminus J)$. Since I is an independent set

$$\mu(I \cap J) \leq \mu(J) - \mu(J \cap \partial(I \cap J_{2,n})).$$

Note that $\mu(\partial(I \cap J_{2,n}) \cap J_{i,n})$ is 0 if $i = 2$ but is $\frac{1}{t-1}\mu(\partial(I \cap J_{2,n}))$ otherwise (see the proof of Theorem 3 for more explanation). Thus, by Theorem 1,

$$(19) \quad \mu(I \cap J) \leq \mu(J) - \frac{1}{t-1}\mu(\partial(I \cap J_{2,n}))$$

$$(20) \quad \leq \frac{1}{t} - \frac{1}{t-1}\Phi_t\left(\frac{\delta}{t-1}\right)$$

$$(21) \quad \leq \frac{1}{t} - \frac{1}{t-1}\left(\frac{\delta}{t-1}\right)^{1/\eta(t)}.$$

Since $\mu(I) > \frac{3t-2}{t^3}$, we have that

$$\frac{1}{t} + \delta - \frac{1}{t-1}\left(\frac{\delta}{t-1}\right)^{1/\eta(t)} > \frac{3t-2}{t^3}.$$

Thus, we obtain that

$$(22) \quad \frac{(t-2)(t-1)}{t^2} > \left(\frac{t\delta}{t-1}\right)^{1/\eta(t)} - t\delta.$$

Note that the two sides of the inequality are equal at $\delta_j = \frac{t-1}{t^4}$ and $\delta_j = \frac{t-1}{t^3}$. Note that since $1/\eta(t) \in (0, 1)$ for all $t \geq 3$, the RHS of (22) is concave for all $\delta \geq 0$. Thus, (22) is false when $\delta \in [\frac{t-1}{t^4}, \frac{t-1}{t^3}]$. Therefore, we have (18). \square

From Theorem 10, we can attain a bound that is even better.

Claim 14. *Let $I \subset [t]^n$ be a sorted independent set such that $\mu(I) > \frac{3t-2}{t^3}$, then for all $j \in [n]$,*

$$(23) \quad \mu(I \setminus J_{1,j}) < \frac{t-1}{t^4} \text{ or } \mu(I \setminus J_{1,j}) > \frac{(2t-1)(t-1)}{t^4}.$$

Proof. Again, we may assume without loss of generality that $j = n$, let $J = J_{1,j}$. Let $\delta = \mu(I \setminus J)$. From Claim 13, we only need to consider the case that

$$(24) \quad \frac{(2t-1)(t-1)}{t^4} \geq \delta > \frac{t-1}{t^3}.$$

From (20)

$$\mu(I \cap J) \leq \frac{1}{t} - \frac{1}{t-1}\Phi_t\left(\frac{\delta}{t-1}\right).$$

Now make the substitution

$$\delta = \frac{(t-1)}{t^3}(1 + \delta'),$$

where $\delta' \in (0, \frac{t-1}{t}]$. From Theorem 10,

$$\begin{aligned} \Phi_t\left(\frac{\delta}{t-1}\right) &= \Phi_t\left(\frac{1+\delta'}{(t-1)^3}\right) \\ &= \frac{(t-1)^2}{t^2} \Phi_t\left(\frac{1+\delta'}{t}\right) \\ &= \frac{(t-1)^2}{t^2} \left(\frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{\delta'}{t-1}\right)\right) \\ &\geq \frac{(t-1)^2}{t^2} \left(\frac{t-1}{t} + \frac{1}{t} \left(\frac{\delta'}{t-1}\right)^{1/\eta(t)}\right). \end{aligned}$$

Hence, since $\mu(I) > \frac{3t-2}{t^3}$,

$$\frac{t-1}{t^3}(1+\delta') + \frac{1}{t} - \frac{t-1}{t^2} \left(\frac{t-1}{t} + \frac{1}{t} \left(\frac{\delta'}{t-1}\right)^{1/\eta(t)}\right) > \frac{3t-2}{t^3}.$$

Rearranging,

$$0 > \left(\frac{\delta'}{t-1}\right)^{1/\eta(t)} - \delta'.$$

Like in the proof of Claim 13, we have equality when $\delta' = 0$ and $\delta' = \frac{t-1}{t}$. Furthermore, since $1/\eta(t) \in (0, 1)$ for all $t \geq 3$, the RHS is concave when $\delta' \geq 0$. Thus, the inequality is false for all $\delta \in (0, \frac{t-1}{t}]$. Therefore, (24) can never hold, proving (23), as desired. \square

The next key step is to show Theorem 4 essentially holds for *compressed* independent sets I .

Lemma 15. *Let $I \subset [t]^n$ be a compressed independent set such that $\mu(I) > \frac{3t-2}{t^3}$, then for some $j \in [n]$,*

$$(25) \quad \mu(I \setminus J_{1,j}) < \frac{t-1}{t^4}.$$

Note that by Lemma 11, we immediately have that Theorem 4 holds for compressed independent sets.

Proof. We prove this statement by induction on n . If $n = 1$, then the bound holds since $I = \{(1)\}$ which is clearly a maximum-sized independent set. Now assume $n \geq 2$ and that the (25) holds for all compressed independent sets $I \subset [t]^{n-1}$ with $\mu(I) > \frac{t-1}{t^3}$.

Fix a compressed independent set $I \subseteq [t]^n$ with $\mu(I) \geq \frac{t-1}{t^3}$. From Claim 14, if the lemma is false, then we have that for all $j \in [n]$,

$$\mu(I \setminus J_{1,j}) > \frac{(2t-1)(t-1)}{t^4}.$$

Since I is compressed, this implies that for all such j

$$\mu(I \cap J_{2,j}) > \frac{2t-1}{t^4}.$$

Recall that for all $a \in [t]$, $I_a = \{(x_1, \dots, x_{n-1}) : (x_1, \dots, x_{n-1}, a) \in I\} \subseteq [t]^{n-1}$. We claim that I_2 is an independent set of K_t^{n-1} . Note that in general I_1 is *not* an independent set of K_t^{n-1} . Since I is compressed, $I_2 \subseteq I_1$. Thus, if there were $x, y \in I_2$ which form an edge of K_t^{n-1} , then $(x, 1), (y, 2) \in I$ form an edge of K_t^n , contradicting that I is an independent set. Therefore, $I_2 \subseteq [t]^{n-1}$ is indeed an independent set.

Note that $\mu(I_2) = t\mu(I \cap J_{2,n}) > \frac{2t-1}{t^3}$ which is not sharp enough of a lower bound to invoke the inductive hypothesis. But, we claim that we can find a compressed independent set $\tilde{I} \subseteq I_1$ such that $\mu(\tilde{I}) \geq \mu(I) > \frac{3t-2}{t^3}$.

Pick $a \in [t]$ such that $(I_1 \setminus I_2) \cap J_{a,n-1} \subseteq [t]^{n-1}$ has maximal size.² Note that since $I_1 \setminus I_2$ is not necessarily compressed, a might not equal 1. Let $\hat{I} = I_2 \cup ((I_1 \setminus I_2) \cap J_{a,n-1})$. We claim that \hat{I} is an independent set (although it might not be compressed). As previously established I_2 is an independent set and clearly $(I_1 \setminus I_2) \cap J_{a,n-1}$ is an independent set since the last coordinate is constant. Thus, if \hat{I} were not an independent set then, there is $x \in I_2$ and $y \in I_1 \setminus I_2$ which are connected by an edge in K_t^{n-1} . But, note that $(x, 2), (y, 1) \in \hat{I}$ are connected by an edge in K_t^n , contradiction. Thus, \hat{I} is an independent set of K_t^{n-1} .

Let \tilde{I} be a compression of \hat{I} . since I_2 and I_1 are already compressed and $I_2 \subseteq \hat{I} \subseteq I_1$, we have that $I_2 \subseteq \tilde{I} \subseteq I_1$. Now,

$$\begin{aligned} \mu(\tilde{I}) &= \mu(\hat{I}) \\ &\geq \mu(I_2) + \frac{\mu(I_1) - \mu(I_2)}{t} \\ &= \frac{\mu(I_1) + (t-1)\mu(I_2)}{t} \\ &\geq \frac{1}{t} \sum_{i=1}^t \mu(I_i) \\ &= \mu(I) > \frac{3t-2}{t^3}. \end{aligned}$$

Thus, we may now invoke the induction hypothesis on \tilde{I} . Therefore, there exists $j \in [n-1]$ such that

$$\mu(\tilde{I} \setminus J_{1,j}) < \frac{t-1}{t^4}.$$

Since $I_2 \subseteq \tilde{I}$, we have that

$$\mu(I_2 \setminus J_{1,j}) \leq \mu(\tilde{I} \setminus J_{1,j}) < \frac{t-1}{t^4}.$$

Therefore, since I is compressed

$$(26) \quad \mu(I \setminus (J_{1,j} \cup J_{1,n})) = \frac{1}{t} \sum_{i=2}^n \mu(I_i \setminus J_{1,j})$$

$$(27) \quad \leq \frac{t-1}{t} \mu(I_2 \setminus J_{1,j})$$

$$(28) \quad \leq \frac{(t-1)^2}{t^5}.$$

Hence, recalling that I is very far from $J_{1,n}$

$$(29) \quad \mu((I \setminus J_{1,n}) \cap J_{1,j}) = \mu(I \setminus J_{1,n}) - \mu(I \setminus (J_{1,j} \cup J_{1,n}))$$

$$(30) \quad \geq \frac{(2t-1)(t-1)}{t^4} - \frac{(t-1)^2}{t^5} = \frac{(2t^2 - 2t + 1)(t-1)}{t^5}.$$

²To keep notation as concise as possible, we use the $J_{i,j}$ notation to refer to both the maximal independent sets of $[t]^{n-1}$ and $[t]^n$. It should be clear from context which we are referring to.

Likewise,

$$(31) \quad \mu((I \setminus J_{1,j}) \cap J_{1,n}) = \mu(I \setminus J_{1,j}) - \mu(I \setminus (J_{1,j} \cup J_{1,n}))$$

$$(32) \quad \geq \frac{(2t-1)(t-1)}{t^4} - \frac{(t-1)^2}{t^5} = \frac{(2t^2-2t+1)(t-1)}{t^5}.$$

Let $I' = I \cap J_{2,j} \cap J_{1,n}$ and $I'' = I \cap J_{1,j} \cap J_{2,n}$. Now observe that since I is compressed

$$\mu(I') = \mu(I \cap J_{2,j} \cap J_{1,n}) \geq \frac{1}{t-1} \mu((I \setminus J_{1,j}) \cap J_{1,n}) = \frac{2t^2-2t+1}{t^5}.$$

Similarly,

$$\mu(I'') = \mu(I \cap J_{1,j} \cap J_{2,n}) \geq \frac{1}{t-1} \mu((I \setminus J_{1,n}) \cap J_{1,j}) = \frac{2t^2-2t+1}{t^5}.$$

Since I' is constant in both the j th and n th coordinates,

$$\mu(\partial I' \cap J_{1,j} \cap J_{2,n}) = \frac{1}{(t-1)^2} \mu(\partial I') \geq \frac{1}{(t-1)^2} \Phi_t(\mu(I')).$$

From Theorem 10, we have that

$$\begin{aligned} \Phi_t(\mu(I')) &\geq \Phi_t\left(\frac{1}{t^3} + \frac{(t-1)^2}{t^5}\right) \\ &= \frac{(t-1)^2}{t^2} \left(\frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{t-1}{t^2}\right)\right) \\ &\geq \frac{(t-1)^3}{t^3} \end{aligned}$$

since $\Phi_t(\nu) \geq 0$. Therefore, since $I' \cup I''$ is an independent set

$$\begin{aligned} \frac{1}{t^2} &= \mu(J_{1,j} \cap J_{2,n}) \\ &\geq \mu(I'') + \mu(\partial I' \cap J_{1,j} \cap J_{2,n}) \\ &\geq \frac{2t^2-2t+1}{t^5} + \frac{1}{(t-1)^2} \Phi_t(\mu(I')) \\ &\geq \frac{2t^2-2t+1}{t^5} + \frac{t-1}{t^3} \\ &= \frac{t^3+t^2-2t+1}{t^5} > \frac{1}{t^2}, \text{ (since } t \geq 3) \end{aligned}$$

contradiction. Thus, the lemma is true. \square

Now we extend this result to sorted independent sets; and thus all independent sets.

Lemma 16. *Let $I \subset [t]^n$ be a sorted independent set such that $\mu(I) > \frac{3t-2}{t^3}$, then for some $j \in [n]$,*

$$(33) \quad \mu(I \setminus J_{1,j}) < \frac{t-1}{t^4}.$$

Proof. Like in the proof of Lemma 15, by Claim 14, we may assume for sake of contradiction that for all $j \in [n]$,

$$\mu(I \setminus J_{1,j}) > \frac{(2t-1)(t-1)}{t^4}.$$

It is not hard to see that for all $i, j \in [n]$ such that $i \neq j$,

$$(34) \quad \mu(c_i(I) \setminus J_{1,j}) = \mu(I \setminus J_{1,j}) > \frac{(2t-1)(t-1)}{t^4}.$$

We seek to show that for all $j \in [n]$,

$$(35) \quad \mu(c_j(I) \setminus J_{1,j}) > \frac{(2t-1)(t-1)}{t^4}.$$

By Claim 14, assume for sake of contradiction that

$$(36) \quad \mu(c_j(I) \setminus J_{1,j}) < \frac{t-1}{t^4}$$

for some $j \in [n]$. We may assume without loss of generality that $j = n$. Since I is sorted,

$$(37) \quad \mu(I \cap J_{2,n}) \geq \frac{1}{t-1} \mu(I \setminus J_{1,n}) > \frac{2t-1}{t^4}.$$

Therefore,

$$\mu(\partial(I \cap J_{2,n})) \geq \Phi_t \left(\frac{2t-1}{t^4} \right) = \frac{(t+1)(t-1)^3}{t^4}.$$

This implies that

$$\mu(\partial(I \cap J_{2,n}) \cap J_{1,n}) = \frac{1}{t-1} \mu(\partial(I \cap J_{2,n})) = \frac{(t+1)(t-1)^2}{t^4}.$$

Observe that since I is an independent set

$$\mu(\partial(I \cap J_{2,n}) \cap I) = 0.$$

Therefore, if $x \in \partial(I \cap J_{2,n}) \cap c_n(I)$, then $(x_1, \dots, x_{n-1}, 1) \in I$ (because any other choice for the last coordinate would violate the above relation). Therefore,

$$(38) \quad \mu(\partial(I \cap J_{2,n}) \cap J_{1,n} \cap c_n(I)) \leq \mu(I \cap J_{2,n}).$$

From this, we get that

$$(39) \quad \mu(J_{1,n} \setminus c_n(I)) \geq \mu((\partial(I \cap J_{2,n}) \cap J_{1,n}) \setminus c_n(I))$$

$$(40) \quad = \mu((\partial(I \cap J_{2,n}) \cap J_{1,n})) - \mu((\partial(I \cap J_{2,n}) \cap J_{1,n}) \cap c_n(I))$$

$$(41) \quad \geq \mu((\partial(I \cap J_{2,n}) \cap J_{1,n})) - \mu(I \cap J_{2,n}) \text{ (by (38))}$$

Next, we deduce

$$(42) \quad \mu(I) = \mu(c_n(I) \cap J_{1,n}) + \mu(c_n(I) \setminus J_{1,n})$$

$$(43) \quad < \frac{1}{t} - \mu(J_{1,n} \setminus c_n(I)) + \frac{t-1}{t^4} \text{ (by (36))}$$

$$(44) \quad \leq \frac{1}{t} - (\mu(\partial(I \cap J_{2,n}) \cap J_{1,n}) - \mu(I \cap J_{2,n})) + \frac{t-1}{t^4} \text{ (by (41))}$$

Let $\nu := \mu(I \cap J_{2,n})$. Then note that

$$\mu(\partial(I \cap J_{2,n}) \cap J_{1,n}) = \frac{1}{t-1} \mu(\partial(I \cap J_{2,n})) \geq \frac{1}{t-1} \Phi_t(\nu).$$

Thus, by (44)

$$(45) \quad \mu(I) < \frac{t^3 + t - 1}{t^4} - \left(\frac{1}{t-1} \Phi_t(\nu) - \nu \right).$$

We divide the remainder of the proof into three cases depending on the value of ν .

Case³ 1: $\frac{2t-1}{t^4} < \nu \leq \frac{1}{t^2}$. By Theorem 10 and the fact that $\Phi_t(\rho) \geq \rho$ for all $\rho \in [0, 1]$,

$$\begin{aligned}\Phi_t(\nu) &= \frac{(t-1)^2}{t^2} \left(\frac{t-1}{t} + \frac{1}{t} \Phi_t \left(\frac{t^3\nu - 1}{t-1} \right) \right) \\ &= \frac{(t-1)^2}{t^2} \left(\frac{t-1}{t} + \frac{1}{t} \left(\frac{t-1}{t} + \frac{1}{t} \Phi_t \left(\frac{t^4\nu - (2t-1)}{(t-1)^2} \right) \right) \right) \\ &\geq \frac{(t-1)^3(t+1) + t^4\nu - (2t-1)}{t^4}.\end{aligned}$$

Thus, by (45)

$$\frac{3t-2}{t^3} < \mu(I) < \frac{t^3+t-1}{t^4} - \frac{1}{t-1} \cdot \frac{(t-1)^3(t+1) + t^4\nu - (2t-1)}{t^4} + \nu.$$

Rearranging,

$$\frac{(2t-1) + 2(t-1)^3}{(t-1)t^4} \leq \left(\frac{t-1}{t-2} \right) \nu \leq \frac{t-2}{t^2(t-1)}.$$

This implies that

$$2(t-1)^3 + 2t - 1 < t^3 - 2t^2.$$

Thus, $t^3 - 4t^2 + 8t - 2 < 0$, but this is false for $t \geq 3$, contradiction.

Case 2, $\frac{1}{t^2} < \nu \leq \frac{(2t-1)(t-1)}{t^4}$.

Then $\Phi_t(\nu) \geq \frac{(t-1)^2}{t^2}$. Thus, by (45)

$$\mu(I) < \frac{t^3+t-1}{t^4} - \frac{t-1}{t^2} + \frac{(2t-1)(t-1)}{t^4} = \frac{3t^2-2t}{t^4} = \frac{3t-2}{t^4} < \mu(I),$$

contradiction.

Case 3, $\nu > \frac{(2t-1)(t-1)}{t^4}$.

Observe that

$$\begin{aligned}\Phi_t(\nu) &\geq \Phi_t \left(\frac{2t^2 - 3t + 1}{t^4} \right) = \frac{t-1}{t} \Phi_t \left(\frac{2t^2 - 3t + 1}{t^3} \right) \\ &= \frac{(t-1)^2}{t^2} + \frac{t-1}{t^2} \Phi_t \left(\frac{t^2 - 3t + 1}{t(t-1)} \right) \\ &\geq \frac{t(t-1)^2 + (t^2 - 3t + 1)}{t^3} \\ &> \frac{t^2(t-1)^2 + (t-1)(t^2 - 3t + 1)}{t^4} \quad (\text{since } t \geq 3) \\ &= \frac{(t-1)(t^3 - 3t + 1)}{t^4}.\end{aligned}$$

Since I is sorted, $\mu(I) \geq 2\nu$. Therefore,

$$2\nu \leq \mu(I) < \frac{t^3+t-1}{t^4} - \frac{t^3-3t+1}{t^4} + \nu.$$

Thus, $\nu < \frac{4t-2}{t^4}$, but $\frac{4t-2}{t^4} \leq \frac{(2t-1)(t-1)}{t^4}$ for $t \geq 3$, contradiction.

End Cases.

Therefore, our assumption that (35) failed to hold is false. Therefore

$$\mu(c_j(I) \setminus J_{1,i}) > \frac{(2t-1)(t-1)}{t^4}.$$

³Recall that $\nu > \frac{2t-1}{t^4}$ by (37)

for all $i, j \in [n]$. Applying this fact repeatedly, we can find a compressed I' of the same cardinality as I such that $\mu(I' \setminus J_{1,i}) > \frac{(2t-1)(t-1)}{t^4}$ for all $i \in [n]$, contradicting Lemma 15. Thus, our counterexample I could have never existed. This proves the Lemma. \square

Proof of Theorem 4. Let $I \subset [t]^n$ be an independent set with $\mu(I) > \frac{3t-2}{t^3}$. Assume without loss of generality that I is sorted. By Lemma 16, we know that there is $j \in [n]$ such that

$$\mu(I \setminus J_{1,j}) \leq \frac{t-1}{t^4} < \frac{1}{t^3}.$$

Thus, by Lemma 11, we have that

$$\mu(I \setminus J_{1,j}) \leq 4\epsilon^{\eta(t)},$$

as desired. \square

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REFERENCES

- [ADFS04] N. Alon, I. Dinur, E. Friedgut, and B. Sudakov. Graph Products, Fourier Analysis and Spectral Techniques. *Geometric & Functional Analysis GAFA*, 14(5):913–940, 2004.
- [AS04] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. John Wiley & Sons, April 2004. Google-Books-ID: q3lUjheWiMoC.
- [BG16] Joshua Brakensiek and Venkatesan Guruswami. New Hardness Results for Graph and Hypergraph Colorings. In Ran Raz, editor, *31st Conference on Computational Complexity (CCC 2016)*, volume 50 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 14:1–14:27, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [BHT00] S. Bobkov, C. Houdré, and P. Tetali. λ_∞ , Vertex Isoperimetry and Concentration. *Combinatorica*, 20(2):153–172, February 2000.
- [BM08] József Balogh and Dhruv Mubayi. A new short proof of a theorem of Ahlswede and Khachatrian. *Journal of Combinatorial Theory, Series A*, 115(2):326–330, February 2008.
- [CEK13] Demetres Christofides, David Ellis, and Peter Keevash. An Approximate Vertex-Isoperimetric Inequality for r -sets. *The Electronic Journal of Combinatorics*, 20(4):P15, August 2013.
- [DF09] Irit Dinur and Ehud Friedgut. Intersecting Families Are Essentially Contained in Juntas. *Comb. Probab. Comput.*, 18(1-2):107–122, March 2009.
- [DFR08] Irit Dinur, Ehud Friedgut, and Oded Regev. Independent Sets in Graph Powers are Almost Contained in Juntas. *Geometric and Functional Analysis*, 18(1):77–97, April 2008.
- [DS05] Irit Dinur and Samuel Safra. On the Hardness of Approximating Minimum Vertex Cover. *Annals of Mathematics*, 162(1):439–485, 2005.
- [EKL16a] David Ellis, Nathan Keller, and Noam Lifshitz. On the structure of subsets of the discrete cube with small edge boundary. *arXiv:1612.06680 [math]*, December 2016. arXiv: 1612.06680.
- [EKL16b] David Ellis, Nathan Keller, and Noam Lifshitz. Stability versions of Erdős-Ko-Rado type theorems, via isoperimetry. *arXiv:1604.02160 [math]*, April 2016. arXiv: 1604.02160.
- [EKL17] David Ellis, Nathan Keller, and Noam Lifshitz. On a Biased Edge Isoperimetric Inequality for the Discrete Cube. *arXiv:1702.01675 [math]*, February 2017. arXiv: 1702.01675.
- [EKN17] David Ellis, Gil Kalai, and Bhargav Narayanan. On symmetric intersecting families. *arXiv:1702.02607 [math]*, February 2017. arXiv: 1702.02607.
- [EL16] David Ellis and Noam Lifshitz. On the union of intersecting families. *arXiv:1610.03027 [math]*, October 2016. arXiv: 1610.03027.
- [Fil16] Yuval Filmus. Ahlswede-Khachatrian Theorems: Weighted, Infinite, and Hamming. *arXiv:1610.00756 [math]*, October 2016. arXiv: 1610.00756.

- [FKMW16] Yuval Filmus, Guy Kindler, Elchanan Mossel, and Karl Wimmer. Invariance Principle on the Slice. In Ran Raz, editor, *31st Conference on Computational Complexity (CCC 2016)*, volume 50 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 15:1–15:10, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [FM16] Yuval Filmus and Elchanan Mossel. Harmonicity and Invariance on Slices of the Boolean Cube. In Ran Raz, editor, *31st Conference on Computational Complexity (CCC 2016)*, volume 50 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:13, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [Fri08] Ehud Friedgut. On the measure of intersecting families, uniqueness and stability. *Combinatorica*, 28(5):503–528, September 2008.
- [GH08] Mahya Ghandehari and Hamed Hatami. Fourier analysis and large independent sets in powers of complete graphs. *Journal of Combinatorial Theory, Series B*, 98(1):164–172, January 2008.
- [GL74] D. Greenwell and L. Lovász. Applications of product colouring. *Acta Mathematica Academiae Scientiarum Hungarica*, 25(3-4):335–340, September 1974.
- [HBP14] Andy Hammerlindl, John Bowman, and Tom Prince. Asymptote: The vector graphics language, 2014.
- [Hun07] John D Hunter. Matplotlib: A 2d Graphics Environment. *Computing in Science & Engineering*, 9(3):90–95, May 2007.
- [Kee08] Peter Keevash. Shadows and intersections: Stability and new proofs. *Advances in Mathematics*, 218(5):1685–1703, August 2008.
- [KL16a] Nathan Keller and Noam Lifshitz. On Large H-Intersecting Families. *arXiv:1609.01884 [math]*, September 2016. arXiv: 1609.01884.
- [KL16b] Nathan Keller and Noam Lifshitz. A tight stability version of the Complete Intersection Theorem. *arXiv:1604.06135 [math]*, April 2016. arXiv: 1604.06135.
- [KM10] Peter Keevash and Dhruv Mubayi. Set systems without a simplex or a cluster. *Combinatorica*, 30(2):175–200, March 2010.
- [Sau72] N Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, July 1972.
- [She72] Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, April 1972.

APPENDIX A. PROOFS OF ALGEBRAIC INEQUALITIES

Proof of Claim 8. For $c \geq 0$, let $f_c(z) = (z + c)^{\alpha(t)} - z^{\alpha(t)}$. Notice that if $z > 0$, then $f'_c(z) = (\alpha(t))((z + c)^{\alpha(t)-1} - z^{\alpha(t)-1}) \leq 0$. Thus, we have that $(t - 1)f_c(y) \geq (t - 1)f_c(x)$ for all $c \geq 0$. Consider $c = (t - 1)y$; we then have that

$$\begin{aligned} (t - 1)f_c(y) &= (t - 1)((ty)^{\alpha(t)} - y^{\alpha(t)}) = (t - 1)(t^{\alpha(t)} - 1)y^{\alpha(t)} = y^{\alpha(t)} \geq \\ (t - 1)f_c(x) &= (t - 1)((x + (t - 1)y)^{\alpha(t)} - x^{\alpha(t)}). \end{aligned}$$

Rearranging, we obtain (7). □

Proof of Claim 12. First, verify the cases $t = 3$ and $t = 4$ using a calculator. Notice that $\eta(t) = \frac{\log t}{\log t - \log(t-1)} \leq t \log t$ so

$$\left(\frac{(t-1)^3}{(t-1)^3 - t} \right)^{\eta(t)} \leq e^{\frac{t\eta(t)}{(t-1)^3 - t}} \leq e^{\frac{t^2 \log t}{(t-1)^3 - t}}.$$

Also use a calculator to verify that $h(t) := \frac{t^2 \log t}{(t-1)^3 - t}$ is less than 1 for $t = 5$. Now observe that when going from t to $t + 1$, the numerator increases by

$$\begin{aligned} (t + 1)^2 \log(t + 1) - t^2 \log t &= (2t + 1) \log(t + 1) + t^2 \log\left(1 + \frac{1}{t}\right) \\ &\leq (2t + 1) \log(t + 1) + t \leq (2t + 1)t + t \\ &= 2t^2 + 2t. \end{aligned}$$

and the denominator increases by

$$t^3 - (t + 1) - (t - 1)^3 + t = 3t^2 - 3t$$

Since $2t^2 + 2t \leq 3t^2 - 3t$ for all $t \geq 5$ and $h(5) \leq 1$, we have by a simple inductive proof that $h(t) \leq 1$ for all $t \geq 5$. Thus, for all $t \geq 5$,

$$\left(\frac{(t-1)^3}{(t-1)^3 - t} \right)^{\eta(t)} \leq e^1 < 4,$$

as desired. \square

APPENDIX B. PROOF OF THEOREM 10

The first step in proving this theorem is to determine the structure of S when $\mu(S)$ is fixed but $\mu(\partial S)$ is minimized. In particular, we need S to look as much like a maximal independent set (e.g., $J = [t]^{n-1} \times [1]$) as possible.

Claim 17. *Let $t \geq 3$ and n be positive integers. Let J be a maximum-sized independent set. Consider $S \subseteq [t]^n$.*

- (1) *If $\mu(S) < \frac{1}{t}$, then there exists $S' \subset [t]^n$ such that $\mu(S') = \mu(S)$, $\mu(\partial S') \leq \mu(\partial S)$, and $S' \subset J$.*
- (2) *If $\mu(S) \geq \frac{1}{t}$, then there exists $S' \subset [t]^n$ such that $\mu(S') = \mu(S)$, $\mu(\partial S') \leq \mu(\partial S)$, and $J \subseteq S'$.*

For each $x \in S$, define $|x|$, the *level* of x , be the number of coordinates of x not equal to 1 (c.f., [ADFS04]).

Proof. Without loss of generality, assume that $J = [t]^{n-1} \times [1]$. By Claim 6, we may assume that S is compressed. This immediately resolves the case $n = 1$, so we may assume $n \geq 2$.

Consider the map $\Pi : [t]^n \rightarrow \{0, 1\}^n$ such that

$$(46) \quad \Pi(x)_i := \begin{cases} 0 & x_i = 1 \\ 1 & x_i \neq 1 \end{cases} \text{ for all } i \in [n].$$

Let $f_S := \mathbf{1}_{\Pi(S)} : \{0, 1\}^n \rightarrow \{0, 1\}$ be the indicator function of $\Pi(S)$. Since S is compressed, f_S is a *monotone* Boolean function: $f_S(x) \leq f_S(y)$ whenever $x_i \leq y_i$ for all $i \in [n]$.

For all $z \in \{0, 1\}^n$, let $\neg z$ denote the bitwise complement of z . Note that for any $x \in \Pi^{-1}(z)$ and $y \in \Pi^{-1}(z)$, x and y are connected by an edge in K_t^n . Therefore, because S is compressed

$$(47) \quad \partial S = \bigcup_{z \in \Pi(S)} \Pi^{-1}(\neg z)$$

and so

$$(48) \quad \mu(\partial S) = \frac{1}{t^n} \sum_{z \in \Pi(S)} |\Pi^{-1}(\neg z)| = \frac{1}{t^n} \sum_{z \in \Pi(S)} (t-1)^{n-|z|}.$$

We now describe an algorithm which modifies S into a compressed S' such that $\mu(S' \cap J)$ is maximized while keeping $\mu(\partial S') \leq \mu(\partial S)$ and $\mu(S) \leq \mu(S')$. This algorithm consists of two subroutines.

Filling. See Figure 3. Let

$$\text{fill}(S) = \bigcup_{z \in \Pi(S)} \Pi^{-1}(z).$$

Note that $S \subseteq \text{fill}(S)$ but $\Pi(S) = \Pi(\text{fill}(S))$, so $\mu(\partial(\text{fill}(S))) = \mu(\partial S)$ by (48).

Note that $\text{fill}(S)$ is compressed since $\mathbf{1}_{\Pi(S)}$ is monotone.

*Folding.*⁴ Assume that $S = \text{fill}(S)$. That is, for each $z \in \Pi(S)$, $\Pi^{-1}(z) \subseteq S$.

⁴Note that this Folding operation is considered another form of compression in the literature, although typically used for Kneser graphs. For example see <https://gilkalai.wordpress.com/2008/10/06/extremal-combinatorics-iv-shifting/>

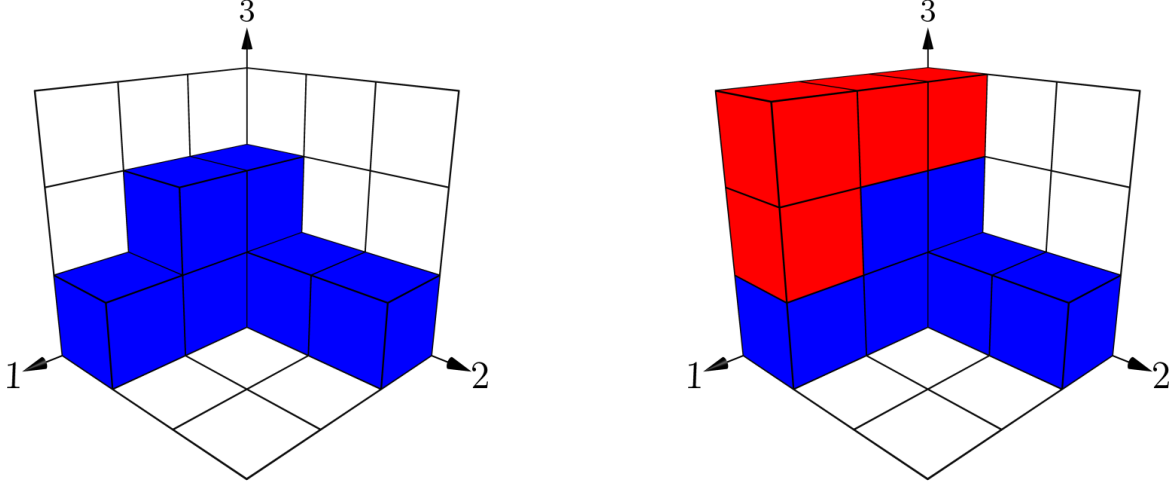


FIGURE 3. A visualization of the operation $\text{fill}(S)$ when $n = t = 3$. Each cube represents an element of S , with the red cubes being the ones that are changed. Each axis label represents a coordinate. For example, the red cube in the upper-left-hand corner represents the vertex $(3, 1, 3)$ of K_3^3 .

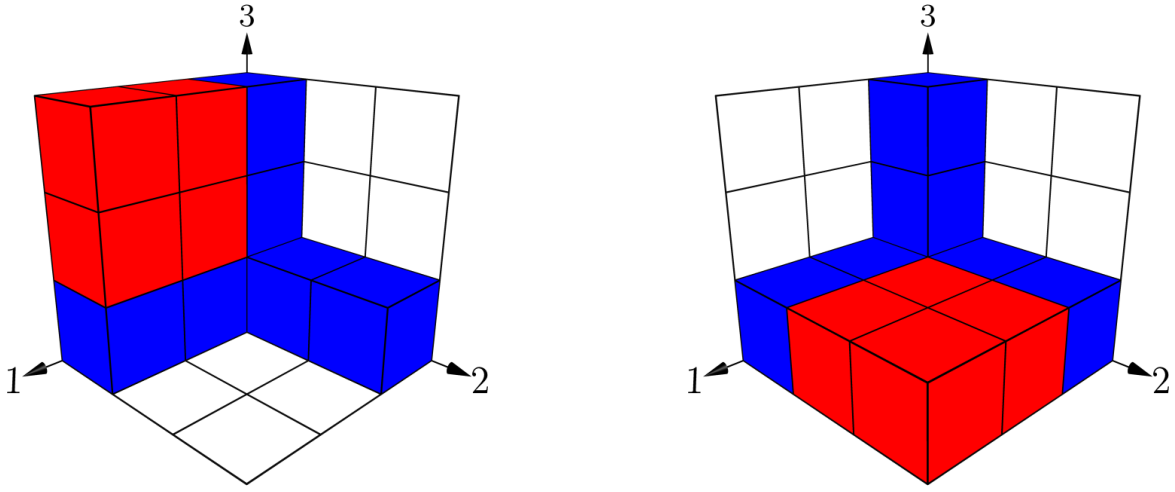


FIGURE 4. A visualization of the operation $\text{fold}_{\{2\}}(S)$ when $n = t = 3$. See the caption for Figure 3 on interpreting this visualization.

The operator fold_A is defined for each subset $A \subseteq [n - 1]$.

For each $B \subseteq [n]$ let $\sigma_B : \{0, 1\}^n \rightarrow \{0, 1\}^n$ be the operator which negates the elements indexed by B

$$\sigma_B(x)_i = \begin{cases} \neg x_i & i \in B \\ x_i & i \notin B \end{cases}.$$

For any $A \subseteq [n - 1]$ let

$$(49) \quad F_A = \{x \in \Pi(S) : x_A = 0, x_n = 1, \sigma_{A \cup \{n\}}(x) \in \Pi(S)\}.$$

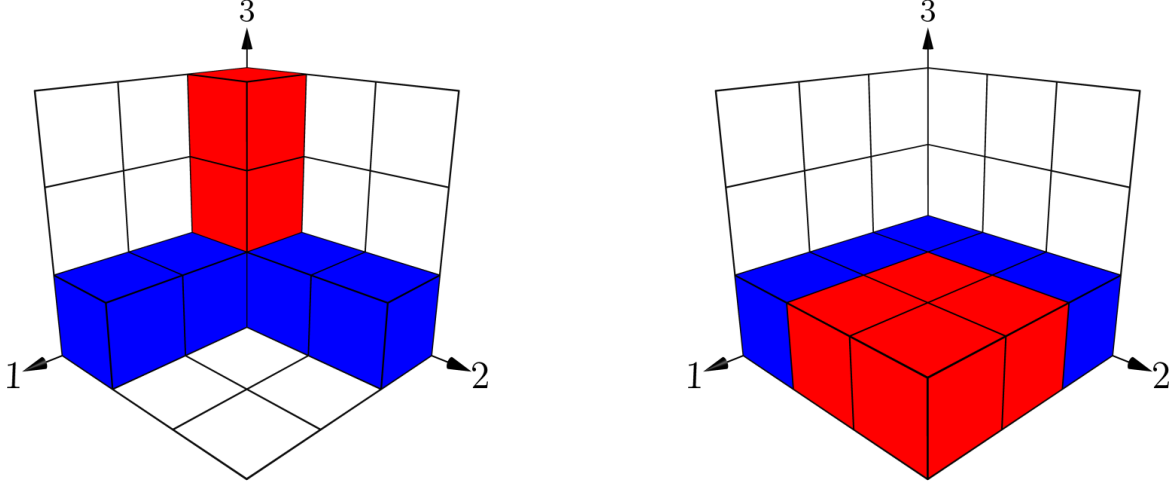


FIGURE 5. A visualization of the operation $\text{fold}_{\{1,2\}}(S)$ when $n = t = 3$. Note that $\mu(S) < \mu(\text{fold}_{\{1,2\}}(S))$. See the caption for Figure 3 on interpreting this visualization.

Then, we define

$$\text{fold}_A(S) := \Pi^{-1}[(\Pi(S) \setminus F_A) \cup \sigma_{A \cup \{n\}}(F_A)].$$

Figures 4 and 5 help to visualize this operator.

First, note that in the case $A = \emptyset$, $F_\emptyset = 0$ since S is compressed. Thus, since $S = \text{fill}(S) = \Pi^{-1}(\Pi(S))$, $\text{fold}_A(S) = S$.

For $A \neq \emptyset$, note that since each element of $x \in \Pi(S)$ either stays the same or is replaced by $y \in \Pi(\text{fold}_A(S))$ such that $|x| \leq |y|$. Thus, since $|\Pi^{-1}(y)| \geq |\Pi^{-1}(x)|$ for all such x and y , we have that $\mu(\text{fold}_i(S)) \geq \mu(S)$. Furthermore, by (48), if we know that $\text{fold}_A(S)$ is compressed, then $\mu(\partial \text{fold}_A(S)) \leq \mu(\partial S)$.

Thus, it suffices to determine when $\text{fold}_A(S)$ is compressed. We claim that this is always the case when $\text{fold}_B(S) = S$ for all $B \subsetneq A$.

Claim 18. *Let $S \subseteq [t]^n$ be compressed and $A \subseteq [n-1]$ nonempty. If $S = \text{fill}(S)$ and $\text{fold}_B(S) = S$ for all $B \subsetneq A$, then $\text{fold}_A(S)$ is compressed and so by the above discussion $\mu(\text{fold}_i(S)) \geq \mu(S)$ and $\mu(\partial \text{fold}_A(S)) \leq \mu(\partial S)$.*

Proof. This is equivalent to showing that $\mathbf{1}_{\Pi(\text{fold}_A(S))} = \mathbf{1}_{(\Pi(S) \setminus F_A) \cup \sigma_{A \cup \{n\}}(F_A)}$ is monotone. Assume for contradiction that there is $x \in \Pi(\text{fold}_A(S))$ and $y \in \{0, 1\}^n \setminus \Pi(\text{fold}_A(S))$ such that $y \leq x$.

First consider the case $x \in \sigma_{A \cup \{n\}}(F_A)$. Thus, $x_i = 1$ for all $i \in A$ and $x_n = 0$. Since $y \leq x$, $y_n = 0$. Let $z = \sigma_{A \cup \{n\}}(x) \in F_A \subseteq \Pi(S)$.

If $y_i = 0$ for some $i \in A$. Then, $y \leq \sigma_{\{i\}}(x)$. Since we assumed $S = \text{fold}_{A \setminus \{i\}}(S)$, we know that $\sigma_{(A \setminus \{i\}) \cup \{n\}}(z) = \sigma_{\{i\}}(x) \in \Pi(S)$. Thus, since S is compressed, $y \in \Pi(S)$. But, $y_n = 0$, so $y \in \Pi(S) \setminus F_A \subseteq \text{fold}_A(S)$, contradiction.

Otherwise, $x \in \Pi(S) \setminus F_A$. Since S is compressed and $y \leq x$, we have that $y \in \Pi(S)$. Thus, since $y \notin \Pi(\text{fold}_A(S))$, we have that $y \in F_A$. Thus $y_n = 1$, so $x_n = 1$. Let $z := \sigma_{A \cup \{n\}}(y) \notin \Pi(S)$.

Let $B \subseteq A$ be the coordinates $i \in B$ for which $x_i = 1$. Then, $z' := \sigma_{B \cup \{n\}}(x)$. Since $x \geq y$, it can be checked that $z' \geq z$. Since S is compressed and $z \notin \Pi(S)$, we have that $z' \notin \Pi(S)$. If $B \subsetneq A$, then this contradicts the fact that $z' \in \Pi(\text{fold}_B(S)) = \Pi(S)$. If $B = A$, then this contradicts the fact that $z' = \sigma_{A \cup \{n\}}(x) \in \Pi(S)$ because $x \notin F_A$. \square

Now that we have defined the operators, we finish the proof. First, set $S' = \text{flat}(S)$. Now, topologically sort the subsets of $[n-1]$ by inclusion. For each $A \subseteq [n-1]$, in this topological order, apply fold_A to S' . If it so happens that applying fold_A causes $\text{fold}_B(S') \neq S'$ for some B earlier in the topological order, we go backtrack to the earliest such B .

By Claim 18, we know that S' is still compressed after each operation. Note that each time S' changes, $\mu(S' \cap J)$ strictly increases. Thus, after some finite number of applications of these operations, we will have a compressed S' such that for all $A \subseteq [n-1]$, $\text{fill}(S') = \text{fold}_A(S') = S'$, $\mu(S') \geq \mu(S)$, and $\mu(\partial S') \leq \mu(\partial S)$.

Furthermore, since $S' = \text{fold}_{[n-1]}(S')$, we know that either $J \subseteq S'$ or $S' \subseteq J$. Now, take any $S'' \subseteq S'$ such that $\mu(S'') = \mu(S)$ while preserving the property that $J \subseteq S''$ or $S'' \subseteq J$. Since $S'' \subseteq S'$, we have that $\mu(\partial S'') \leq \mu(\partial S') \leq \mu(\partial S)$, as desired. \square

With this claim proven, we may now prove the theorem.

Proof of Theorem 10. We divide the proof into four parts.

- *Part 1: If $\nu < \frac{1}{t}$ then $\Phi_t(\nu) \leq \frac{t-1}{t}\Phi_t(t\nu)$.*

Consider any $S \subset [t]^n$ such that $\mu(S) \geq t\nu$. Let

$$S' = S \times [1] \subset [t]^{n+1}$$

be the set where every element of S has a 1 appended. Note that

$$\partial S' = (\partial S) \times \{2, \dots, t\}.$$

Therefore,

$$\begin{aligned} \mu(S') &= \frac{\mu(S)}{t} \geq \nu \\ \mu(\partial S') &= \frac{t-1}{t}\mu(\partial S). \end{aligned}$$

Thus,

$$\frac{t-1}{t}\Phi_t(t\nu) = \inf_{\substack{S \in [t]^* \\ \mu(S) \geq t\nu}} \frac{t-1}{t}\mu(\partial(S)) \geq \inf_{\substack{S' \in [t]^* \\ \mu(S') \geq \nu}} \mu(\partial(S')) = \Phi_t(\nu),$$

where $[t]^* := \bigcup_{n \geq 1} [t]^n$.

- *Part 2: If $\nu < \frac{1}{t}$ then $\Phi_t(\nu) \geq \frac{t-1}{t}\Phi_t(t\nu)$.*

Consider any $S \subset [t]^n$ such that $\mu(S) \geq \nu$. If $n = 1$, then $S = \emptyset$, for which it is trivial that $\Phi_t(0) = 0$. Thus, assume $n \geq 2$.

If $\mu(S) \geq \frac{1}{t}$, then by Theorem 1,

$$\mu(\partial S) \geq \frac{t-1}{t} \geq \frac{t-1}{t}\Phi_t(t\nu).$$

Thus, we may assume $\mu(S) < \frac{1}{t}$. By Claim 17 there is $S' \in [t]^n$ such that $\mu(S') \geq \nu$, $\mu(\partial S') \leq \mu(\partial S)$ and $S' \subseteq [t]^{n-1} \times [1]$. Let

$$S'' = (S')_1 \times [t] = \{(x_1, \dots, x_{n-1}, y) : x \in [t]^n, y \in [t]\} \subset [t]^n.$$

Intuitively, S'' is S' ‘stacked’ t times. Therefore, $\mu(S'') \geq t\nu$. Then,

$$\begin{aligned} \partial S' &= (\partial S')_1 \times \{2, \dots, t\} \\ \partial S'' &= (\partial S')_1 \times \{1, \dots, t\}. \end{aligned}$$

Therefore,

$$\mu(\partial S) \geq \mu(\partial S') = \frac{t-1}{t}\mu(\partial S'') \geq \frac{t-1}{t}\Phi_t(t\nu).$$

Thus,

$$\Phi_t(\nu) = \inf_{\substack{S \in [t]^* \\ \mu(S) \geq \nu}} \mu(\partial S) \geq \frac{t-1}{t} \Phi_t(t\nu).$$

- *Part 3: If $\nu \geq \frac{1}{t}$ then $\Phi_t(\nu) \leq \frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{t\nu-1}{t-1}\right)$.*

For any $S \subseteq [t]^n$ such that $\mu(S) \geq \frac{t\nu-1}{t-1}$, let $S' \subseteq [t]^{n+1}$ be

$$S' := ([t]^n \times [1]) \cup (S \times \{2, \dots, t\}).$$

Then,

$$\partial S' = ([t]^n \times \{2, \dots, t\}) \cup (\partial S \times [1]).$$

Therefore,

$$\begin{aligned} \mu(S') &= \frac{1}{t} + \frac{t-1}{t} \mu(S) \\ \mu(\partial S') &= \frac{t-1}{t} + \frac{1}{t} \mu(\partial S). \end{aligned}$$

Hence,

$$\frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{t\nu-1}{t-1}\right) = \inf_{\substack{S \in [t]^* \\ \mu(S) \geq \frac{t\nu-1}{t-1}}} \left(\frac{t-1}{t} + \frac{1}{t} \mu(\partial S) \right) \geq \inf_{\substack{S' \in [t]^* \\ \mu(S') \geq \nu}} \mu(\partial S') = \Phi_t(\nu).$$

- *Part 4: If $\nu \geq \frac{1}{t}$ then $\Phi_t(\nu) \geq \frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{t\nu-1}{t-1}\right)$.*

For any $S \subseteq [t]^n$ such that $\mu(S) \geq \nu \geq \frac{1}{t}$, by Claim 17 there is $S' \in [t]^n$ such that $\mu(S) = \mu(S')$, $\mu(\partial S') \leq \mu(\partial S)$, and $[t]^{n-1} \times [1] \subseteq S'$. Pick $j \in \{2, \dots, t\}$ such that $\mu(S'_j)$ is maximal.⁵ Then

$$\mu(S'_j) \geq \frac{1}{t-1} \sum_{j=2}^t \mu(S'_j) = \frac{1}{t-1} (t\mu(S') - \mu(S'_1)) = \frac{t\mu(S') - 1}{t-1} \mu(S') \geq \frac{t\nu-1}{t-1}.$$

and also

$$\partial S' \subseteq ([t]^{n-1} \times \{2, \dots, t\}) \cup (\partial S'_j \times [1]).$$

Thus,

$$\mu(\partial S) \geq \mu(\partial S') \geq \frac{t-1}{t} + \frac{1}{t} \mu(\partial S'_j) \geq \frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{t\nu-1}{t-1}\right).$$

Therefore,

$$\Phi_t(\nu) = \inf_{\substack{S \in [t]^* \\ \mu(S) \geq \nu}} \mu(\partial S) \geq \frac{t-1}{t} + \frac{1}{t} \Phi_t\left(\frac{t\nu-1}{t-1}\right).$$

□

⁵Note that we did not define S_j in (9) for $n = 1$. In that case, define S'_j to be \emptyset if $j \notin S'$ and $\{()\}$ if $j \in S'$. It is consistent to define $\mu(\emptyset) = 0$ and $\mu(\{()\}) = 1$.

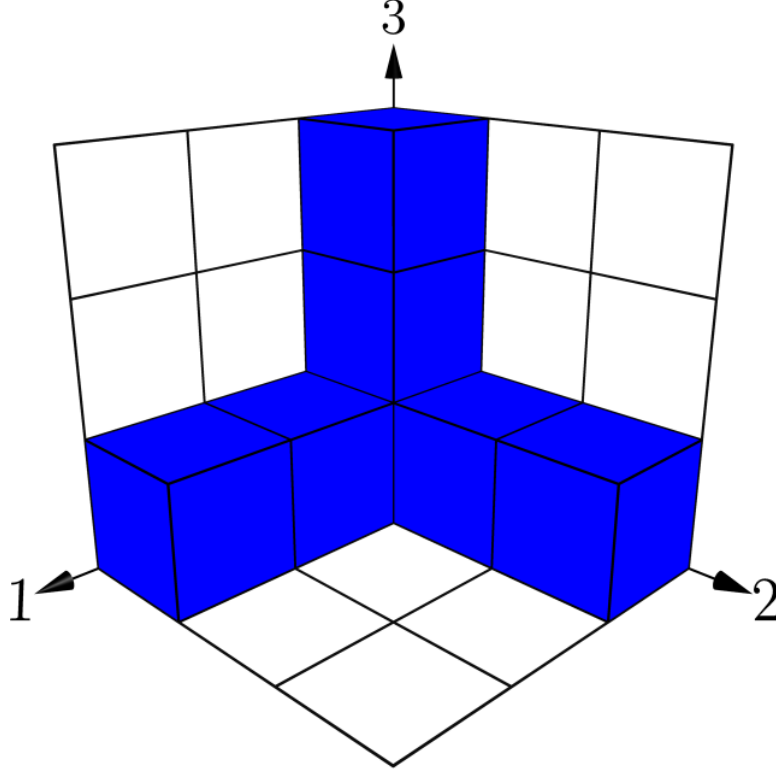


FIGURE 6. Schematic of I_3 when $t = 3$. See the caption for Figure 3 on interpreting this visualization.

APPENDIX C. OPTIMALITY OF EXPONENT IN THEOREM 3

In this appendix, we show in (4) of Theorem 3 that the exponent $\eta(t) = \frac{\log t}{\log t - \log(t-1)}$ is optimal and that the constant factor of 2 is nearly optimal. In other words, the stability result is optimal up to a constant factor.

Lemma 19. *For all $t \geq 3$, there exists an infinite sequence of independent sets $\{I_n\}_{n \geq 3}$ such that $I_n \subset [t]^n$, $\epsilon_n = 1 - t\mu(I_n) > 0$ tends to 0 as $n \rightarrow \infty$, and for any n and any maximum-sized independent set J_n of K_t^n ,*

$$\mu(I_n \setminus J_n) > \frac{t-1}{t} \epsilon^{\eta(t)}.$$

Proof. For $n \geq 3$, consider $J_n = [1] \times [t]^{n-1}$ and

$$(50) \quad I_n := ([t] \times [1]^{n-1}) \cup J_n \setminus ([1] \times \{2, \dots, t-1\}^n)$$

See Figure 6 for a visualization.

One may check that I_n is an independent set of K_t^n and J_n is a maximum-sized independent set which minimizes $\mu(I_n \setminus J_n)$. Furthermore,

$$\mu(I_n) = \frac{t-1}{t^n} + \frac{1}{t} - \frac{(t-1)^{n-1}}{t^n}.$$

Thus,

$$(51) \quad \epsilon_n = \frac{(t-1)^{n-1} - (t-1)}{t^{n-1}}$$

$$(52) \quad \delta_n := \mu(I_n \setminus J_n) = \frac{t-1}{t^n}.$$

Notice that since $t^{1/\eta(t)} = \frac{t-1}{t}$.

$$\begin{aligned} \delta_n^{1/\eta(t)} &= \frac{(t-1)^{1/\eta(t)}}{t^{n/\eta(t)}} \\ &= \left(\frac{t-1}{t}\right)^{1/\eta(t)} \left(\frac{t-1}{t}\right)^{n-1} \\ &= \left(\frac{t-1}{t}\right)^{1/\eta(t)} (\epsilon_n + t\delta_n) \\ &> \left(\frac{t-1}{t}\right)^{1/\eta(t)} \epsilon_n. \end{aligned}$$

Therefore, raising both sides to the $\eta(t)$ power,

$$\delta_n > \frac{t-1}{t} \epsilon_n^{\eta(t)},$$

as desired. □